1.

a.

Note first that $g(\cdot)$ is a function over at least one entire interval, so it must be either a density or a cdf since $g(x) \geq 0 \ \forall x \in \mathbb{R}$. If we can show that $g(\cdot)$ integrates to 1 over $(-\infty, \infty)$, then it must be a pdf.

$$\int_{-\infty}^{\infty} g(x)\,dx = 0 + \int_a^c \frac{2(x-a)}{(b-a)(c-a)}\,dx + 0 + \int_c^b \frac{2(b-x)}{(b-a)(b-c)}\,dx + 0$$

$$= \frac{2}{(b-a)(c-a)} \cdot \left(\frac{x^2}{2} - ax \Big|_a^c \right) + \frac{2}{(b-a)(b-c)} \left(bx - \frac{x^2}{2} \Big|_c^b \right)$$

$$= \frac{c^2 - a^2 - 2(ac - a^2)}{(b-a)(c-a)} + \frac{2(b^2 - bc) - (b^2 - c^2)}{(b-a)(b-c)}$$

$$= \frac{c^2 - 2ac + a^2}{(b-a)(c-a)} + \frac{b^2 - 2bc + c^2}{(b-a)(b-c)}$$

$$= \frac{c-a}{b-a} + \frac{b-c}{b-a}$$

$$= 1.$$

Hence, $g(\cdot)$ is a pdf.

b.

Let X be a random variable with density $g(\cdot)$. Consider the following cases.

If $x < a$, then

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} g(t)\,dt = 0$$

If $a \leq x < c$, then

$$F(x) = P(X \leq x)$$
$$= \int_{-\infty}^{x} g(t)\,dt$$
$$= 0 + \int_a^x \frac{2(t-a)}{(b-a)(c-a)}\,dt$$

$$= \frac{2}{(b-a)(c-a)}\left(\frac{t^2}{2} - at \Big|_a^x\right)$$

$$= \frac{x^2 - a^2 - 2(ax - a^2)}{(b-a)(c-a)}$$

$$= \frac{x^2 - 2ax + a^2}{(b-a)(c-a)}$$

$$= \frac{(x-a)^2}{(b-a)(c-a)}$$

If $x = c$, then

$$F(x) = P(X \le c)$$

$$= \int_{-\infty}^{c} g(t)\,dt$$

$$= 0 + \int_a^c \frac{2(t-a)}{(b-a)(c-a)}\,dt + 0$$

$$= \frac{(c-a)^2}{(b-a)(c-a)}$$

$$= \frac{c-a}{b-a}$$

If $c < x \le b$, then

$$F(x) = P(X \le x)$$

$$= \int_{-\infty}^{x} g(t)\,dt$$

$$= 0 + \int_a^c \frac{2(t-a)}{(b-a)(c-a)}\,dt + \int_c^x \frac{2(b-t)}{(b-a)(b-c)}\,dt$$

$$= \frac{c-a}{b-a} + \frac{2}{(b-a)(b-c)}\left(bt - \frac{t^2}{2} \Big|_c^x\right)$$

$$= \frac{c-a}{b-a} + \frac{2(bx - bc) - (x^2 - c^2)}{(b-a)(b-c)}$$

$$= \frac{c-a}{b-a} + \frac{c^2 + 2bx - 2bc - x^2}{(b-a)(b-c)}$$

$$= \frac{bc - ba - c^2 + ac + c^2 + 2bx - 2bc - x^2}{(b-a)(b-c)}$$

$$= \frac{b^2 - ab - cb + ac - b^2 + 2bx - x^2}{(b-a)(b-c)}$$

$$= \frac{(b-a)(b-c)}{(b-a)(b-c)} - \frac{(b-x)^2}{(b-a)(b-c)}$$

$$= 1 - \frac{(b-x)^2}{(b-a)(b-c)}$$

If $b < x$, then

$$F(x) = P(X \leq x)$$

$$= \int_{-\infty}^{x} g(t)\,dt$$

$$= 0 + \int_a^c \frac{2(t-a)}{(b-a)(c-a)}\,dt + \int_c^b \frac{2(b-t)}{(b-a)(b-c)}\,dt + 0$$

$$= 1 - \frac{(b-b)^2}{(b-a)(b-c)}$$

$$= 1$$

Thus,

$$F(x) = \begin{cases} 0 & \text{if} \quad x < a \\ \dfrac{(x-a)^2}{(b-a)(c-a)} & \text{if} \quad a \leq x < c \\ \dfrac{c-a}{b-a} & \text{if} \quad x = c \\ 1 - \dfrac{(b-x)^2}{(b-a)(b-c)} & \text{if} \quad c < x \leq b \\ 1 & \text{if} \quad b < x \end{cases}$$

2.

Let

$A = \{$"Second king is dealt on the fifth card"$\}$

$B = \{$"Fifth card dealt is a king"$\}$

$C = \{$"One king is dealt in the first four cards"$\}$

Then

$$P(A) = P(B \cap C)$$

$$= P(B|C) \cdot P(C)$$

$$= \underbrace{\frac{3}{52-4}}_{\text{3 kings in the 52-4 cards that remain}} \cdot \underbrace{\frac{\binom{4}{1} \cdot \binom{52-4}{4-1}}{\binom{52}{4}}}_{P(X=1) \text{ for } X \sim \text{hypergeometric}(N=52, n=4, M=4)} \approx .016$$

3.

a. $$m(t) = E(e^{tX})$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{\alpha^x e^{-\alpha}}{x!}$$

$$= e^{-\alpha} e^{\alpha e^t} \cdot \sum_{x=0}^{\infty} \underbrace{\frac{(\alpha e^t)^x e^{-\alpha e^t}}{x!}}_{\text{pmf of the Poisson}(\alpha e^t) \text{ distribution}}$$

$$= e^{\alpha(e^t - 1)} \cdot 1 \quad \text{for any } t \in \mathbb{R}.$$

b. $\frac{d}{dt} m(t) \Big|_{t=0} = \alpha e^t \cdot e^{\alpha(e^t-1)} \Big|_{t=0} = \alpha = E(X)$

$$\frac{d^2}{dt^2} m(t) \Big|_{t=0} = \left[\alpha e^t e^{\alpha(e^t-1)} + (\alpha e^t)^2 e^{\alpha(e^t-1)}\right]\Big|_{t=0}$$

$$= \alpha + \alpha^2$$

$$= Var(X) + E^2(X)$$

$$= E(X^2).$$

4.

a. First observe that $Y = Z^2 \in [0, \infty)$. Then for any $y \geq 0$,

$$F(y) = P(Y \leq y)$$

$$= P(Z^2 \leq y)$$

$$= P(|Z| \leq \sqrt{y})$$

$$= P(-\sqrt{y} \leq Z \leq \sqrt{y})$$

$$= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}),$$

and so

$$f(y) = \frac{d}{dy} F(y)$$

$$= \phi(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - \phi(-\sqrt{y})\left(-\frac{1}{2\sqrt{y}}\right)$$

$$= \frac{1}{2\sqrt{y}}\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{y}} \cdot e^{-\frac{y}{2}} \quad \text{for } y \geq 0.$$

b. $P(Y \leq 9) = P(Z^2 \leq 9) = P(|Z| \leq 3) = P(-3 \leq Z \leq 3) \approx .997.$