# ST 502 MIDTERM 2

November 18, 2019

**NAME:**

**STUDENT ID:**

- You have **75 minutes** to complete this exam.
- This is a **closed book, closed notes** exam. The use of a calculator or computer is NOT permitted.
- Please show all of your work. For this exam, the steps taken to arrive at a particular solution are more important than the final answer.

1. Assume that $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{binomial}(n, p)$. Recall that the mass function for the binomial$(n, p)$ distribution is

$$f(x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x}.$$

(a) (3 points) Give an expression for the likelihood function of this data set.

**Solution:**

$$L(p) = \prod_{i=1}^{n} f(x_i \mid n, p) = \prod_{i=1}^{n} \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i} = p^{n\bar{x}_n} (1-p)^{n(n-\bar{x}_n)} \prod_{i=1}^{n} \binom{n}{x_i}.$$

(b) (3 points) Show that $p \sim \text{beta}(a, b)$ is the conjugate prior for this likelihood. What is the posterior distribution (i.e., the distribution and its parameters)? Recall that the density function for the beta$(a, b)$ distribution is

$$f(x \mid a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1},$$

for $x \in (0, 1)$, $a > 0$, and $b > 0$.

**Solution:**

$$\begin{aligned} \pi(p \mid x_1^n) &\propto p^{n\bar{x}_n} (1-p)^{n(n-\bar{x}_n)} \cdot p^{a-1} (1-p)^{b-1} \\ &= p^{n\bar{x}_n + a - 1} (1-p)^{n(n-\bar{x}_n) + b - 1} \end{aligned}$$

which is the probability density kernel for the beta$\big(n\bar{x}_n + a, n(n-\bar{x}_n) + b\big)$ distribution.

(c) (3 points) Derive the posterior mean of $p$ (i.e., $\mathrm{E}(p \mid x_1^n)$).

**Solution:**

$$\begin{aligned}
\mathrm{E}(p \mid x_1^n) &= \int_0^1 p \cdot \pi(p \mid x_1^n)\ dp \\
&= \int_0^1 p \cdot \frac{\Gamma(a+n^2+b)}{\Gamma(n\bar{x}_n + a)\Gamma\big(n(n-\bar{x}_n)+b\big)} p^{n\bar{x}_n + a - 1}(1-p)^{n(n-\bar{x}_n)+b-1}\ dp \\
&= \frac{\Gamma(a+n^2+b)}{\Gamma(n\bar{x}_n + a)} \frac{\Gamma(n\bar{x}_n + a + 1)}{\Gamma(a+n^2+1+b)} \cdot 1 \\
&= \frac{n\bar{x}_n + a}{a+n^2+b}.
\end{aligned}$$

2. The U-quadratic$(0, \beta)$ distribution is defined for $x \in [0, \beta]$, and has density function

$$f(x \mid \beta) \propto \left(x - \frac{\beta}{2}\right)^2.$$

(a) (3 points) Determine the normalizing constant that makes this a proper density function.

**Solution:**

$$\begin{aligned}
\int_0^\beta \left(x - \frac{\beta}{2}\right)^2\ dx &= \int_0^\beta \left(x - \frac{\beta}{2}\right)^2\ dx \\
&= \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} u^2\ du \\
&= \frac{u^3}{3}\ \Big|_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \\
&= \frac{\beta^3}{12}
\end{aligned}$$

Hence,

$$f(x \mid \beta) = \frac{12}{\beta^3}\left(x - \frac{\beta}{2}\right)^2.$$

(b) (3 points) Suppose that $X_1, \ldots, X_n \overset{\text{iid}}{\sim}$ U-quadratic$(0, \beta)$. Derive the method of moments estimator of $\beta$.

**Solution:** First observe that

$$\begin{aligned}
\mathrm{E}(X) &= \int_0^\beta x \frac{12}{\beta^3}\left(x - \frac{\beta}{2}\right)^2\ dx \\
&= \frac{12}{\beta^3} \int_0^\beta x^3 - 2x^2 \frac{\beta}{2} + x\left(\frac{\beta}{2}\right)^2\ dx \\
&= \frac{12}{\beta^3}\left(\frac{x^4}{4} - x^3 \frac{\beta}{3} + x^2 \frac{\beta^2}{8}\ \Big|_0^\beta\right) \\
&= \frac{\beta}{2}.
\end{aligned}$$

Then the method of moments estimator of $\beta$ is obtained by setting $\bar{x}_n = \mathrm{E}(X)$. Thus,

$$\widehat{\beta}_{mom} = 2\bar{x}_n.$$

3. Suppose that $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{geometric}(p)$. Recall that if $X \sim \text{geometric}(p)$, then $X$ has mass function

$$f(x \mid p) = p(1-p)^{x-1},$$

for $x \in \{1, 2, \ldots\}$, and $\text{E}(X) = \frac{1}{p}$ and $\text{Var}(X) = \frac{1-p}{p^2}$.

(a) (3 points) Derive the MLE of $p$.

**Solution:**

The log-likelihood is

$$l(p) = n\log(p) + n(\bar{x}_n - 1)\log(1-p).$$

Setting the first derivative (with respect to $p$) equal to zero gives,

$$\begin{aligned} \frac{n}{p} - \frac{n(\bar{x}_n - 1)}{1-p} &= 0 \\ n(1-p) &= np(\bar{x}_n - 1) \\ \widehat{p}_{mle} &= \frac{1}{\bar{x}_n}. \end{aligned}$$

(b) (3 points) Prove that the MLE from part (a) is a consistent estimator of $p$. That is, show that $\widehat{p} \xrightarrow{P} p$ as $n \to \infty$. Hint: use the fact that $\bar{X}_n \geq 1$ (since each $X_i \geq 1$).

**Solution:**

Let $\varepsilon > 0$, and observe that

$$\begin{aligned} P\big(|\widehat{p} - p| > \varepsilon\big) &= P\bigg(\Big|\frac{1}{\bar{X}_n} - p\Big| > \varepsilon\bigg) \\ &= P\bigg(\frac{p}{\bar{X}_n}\Big|\frac{1}{p} - \bar{X}_n\Big| > \varepsilon\bigg). \end{aligned}$$

Then since $\bar{X}_n \geq 1$ it follows that $\frac{1}{\bar{X}_n} \leq 1$, and so

$$\begin{aligned} P\big(|\widehat{p} - p| > \varepsilon\big) &\leq P\bigg(p\Big|\frac{1}{p} - \bar{X}_n\Big| > \varepsilon\bigg) \\ &= P\bigg(p^2\Big(\frac{1}{p} - \bar{X}_n\Big)^2 > \varepsilon^2\bigg) \\ &\leq \frac{p^2}{\varepsilon^2}\text{E}\Big[\Big(\frac{1}{p} - \bar{X}_n\Big)^2\Big], \quad \text{by Markov's inequality} \\ &= \frac{p^2}{\varepsilon^2}\text{Var}(\bar{X}_n), \quad \text{since } \text{E}(\bar{X}_n) = \text{E}(X) = \frac{1}{p} \\ &= \frac{p^2}{\varepsilon^2}\frac{1-p}{np^2} \\ &= \frac{1-p}{n\varepsilon^2} \longrightarrow 0 \end{aligned}$$

as $n \to \infty$.

(c) (3 points) Derive the asymptotic variance of the MLE. Recall that

$$\text{I}(p) = -\text{E}\Big(\frac{\partial^2}{\partial p^2}\log\big(f(X \mid p)\big)\Big).$$

**Solution:**

The Fisher Information for the geometric likelihood function is

$$\begin{aligned}
\mathrm{I}_n(p) &= -\mathrm{E}\Big(\frac{\partial^2}{\partial p^2}\log\big(f(X_1^n \mid p)\big)\Big) \\
&= -\mathrm{E}\Big(\frac{\partial}{\partial p}\Big[\frac{n}{p} - \frac{n(\bar{X}_n - 1)}{1-p}\Big]\Big) \\
&= -\mathrm{E}\Big(\frac{-n}{p^2} - \frac{n(\bar{X}_n - 1)}{(1-p)^2}\Big) \\
&= \frac{n}{p^2} + \frac{n(1/p - 1)}{(1-p)^2} \\
&= \frac{n}{p}\Big(\frac{1}{p} + \frac{1}{(1-p)}\Big) \\
&= \frac{n}{p^2(1-p)}.
\end{aligned}$$

Thus, the asymptotic variance of the MLE is

$$\frac{1}{\mathrm{I}_n(p)} = \frac{p^2(1-p)}{n}.$$

(d) (3 points) Provide an approximate $1 - \alpha$ confidence interval for $p$.

**Solution:**

Using the large sampling behavior of the MLE (theorem from class) and the answers to the previous parts, an approximate $1 - \alpha$ confidence interval for $p$ is

$$\left[\frac{1}{\bar{x}_n} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{p^2(1-p)}{n}}, \frac{1}{\bar{x}_n} + z_{1-\frac{\alpha}{2}}\sqrt{\frac{p^2(1-p)}{n}}\right],$$

where $z_{1-\frac{\alpha}{2}}$ is the $1 - \frac{\alpha}{2}$ quantile of the standard normal distribution.