ST 705 MIDTERM

March 2, 2023

NAME:

STUDENT ID:

- You have **75 minutes** to complete this exam.
- This is a **closed book, closed notes** exam.

1. (3 points) Let
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$
Is A diagonalizable? If so, find the eigenvalues and eigenvectors of A.

2. (3 points) Consider the vector space, $P_3(\mathbb{R})$, of polynomials over $\mathbb{R}$ with degree at most 3, and with the inner product $\langle f, g\rangle := \int_{-1}^{1} f(t)g(t)\, dt$. Beginning with the standard basis, $\{1, x, x^2, x^3\}$, construct an orthonormal basis for $P_3(\mathbb{R})$.

3. Consider the least squares line $y = c \cdot t + d$ corresponding to the m observations $(t_1, y_1), \ldots, (t_m, y_m)$.
   (a) (3 points) Show that the normal equations take the form
$$\left\{c\Big(\sum t_i^2\Big) + d\Big(\sum t_i\Big) = \sum t_i y_i\right\} \bigcap \left\{c\Big(\sum t_i\Big) + md = \sum y_i\right\}.$$
   (b) (3 points) Show that the least squares line must pass through the point $(\bar{t}, \bar{y})$, where $\bar{t}$ and $\bar{y}$ are the averages of the t_i and y_i, respectively.

4. (a) (3 points) Show that the eigenvalues of a projection matrix are either zero or one.
   (b) (3 points) Using the result of part (a), but without directly appealing to the fact that the rank of a matrix is equal to its number of nonzero eigenvalues, show that the rank of a projection matrix is equal to its trace.

5. (3 points) Show that if the least squares estimator $\lambda'\widehat{\beta}$ is the same for all solutions $\widehat{\beta}$ to the normal equations, then $\lambda'\beta$ is estimable.