ST 705 Linear models and variance components
Homework problem set 1

January 8, 2024

1. Prove the following theorem. Let V be a vector space and $B = \{u_1, \ldots, u_n\}$ be a subset of V. Then B is a basis if and only if each $v \in V$ can be expressed *uniquely* as

$$v = a_1 u_1 + \cdots + a_n u_n$$

   for some set of scalars $\{a_1, \ldots, a_n\}$.

2. Show that every eigenvalue of a real symmetric matrix is real.

3. Prove that the eigenvalues of an upper triangular matrix M are the diagonal components of M.

4. Prove that a (square) matrix that is both orthogonal and upper triangular must be a diagonal matrix.

5. Let $x = (x_1, \ldots, x_p)' \in \mathbb{R}^p$. Show that for $i \in \{1, \ldots, p\}$,

$$|x_i| \leq \|x\|_2 \leq \|x\|_1,$$

   where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the l_1 and l_2 vector norms, respectively.

6. Prove that all norms on a finite-dimensional vector space V over $\mathbb{C}$ are *equivalent*. That is, show that for any two norms, say $\|\cdot\|_a$ and $\|\cdot\|_b$, defined on V, there exists real-valued positive constants c_1 and c_2 such that for every $x \in V$,

$$c_1\|x\|_b \leq \|x\|_a \leq c_2\|x\|_b.$$

   (a) First, show that it is without loss of generality to consider $\|\cdot\|_b = \|\cdot\|_1$.

   (b) Second, demonstrate that it suffices to only consider $x \in V$ with $\|x\|_1 = 1$.

   (c) Next, prove that any norm $\|\cdot\|_a$ is a continuous function under $\|\cdot\|_1$-distance.

(d) Finally, apply a result from calculus such as the Bolzano-Weierstrass theorem or the extreme value theorem to finish your argument that all norms on a finite-dimensional vector space are *equivalent*.

This notion of *equivalence* is in reference to the fact that if a sequence is convergent in *some* norm, then it is convergent in *all* norms. Note the assumption of a *finite*-dimensional vector space.