ST 705 Linear models and variance components
Homework problem set 2

January 16, 2024

1. Monahan exercise A.50.

2. Let $A \in \mathbb{R}^{p \times p}$ be symmetric. Use the spectral decomposition of A to show that

$$\sup_{x \in \mathbb{R}^p \setminus \{0\}} \frac{x'Ax}{x'x} = \lambda_{\max},$$

   where $\lambda_{\max}$ is the largest eigenvalue of A. Observe that this is a special case of the Courant-Fischer theorem (see `https://en.wikipedia.org/wiki/Min-max_theorem`).

3. Construct an $n \times n$ matrix A such that $\lambda_{\max}(A) \neq \sup_{v \neq 0} \left\{ \frac{v'Av}{v'v} \right\}$, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of its argument. Why does your counter example not violate the Courant-Fischer theorem?

4. Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$.

5. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that AB can be written as a sum of n matrices of rank at most one. Hint: think about empirical covariance matrices.

6. Let V be a finite-dimensional inner product space over $\mathbb{C}$, and let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V.

   (a) Show that for any $x, y \in V$,

$$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

   This is called Parseval's identity.

   (b) For $V = \mathbb{R}^2$ use Parseval's identity to prove the Pythagorean theorem. Generalize this result to $\mathbb{R}^n$.

7. Let V be an inner product space over $\mathbb{C}$, and let $\{v_1, \ldots, v_n\} \subset V$ be orthonormal.

(a) Prove that for any $x \in V$,

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2.$$

This is called Bessel's inequality.

(b) Show that Bessel's inequality is an equality if and only if $x \in \operatorname{span}\{v_1, \ldots, v_n\}$.