ST 705 Linear models and variance components
Homework problem set 3

January 24, 2024

1. Let Σ be a $p \times p$ symmetric non-negative definite matrix, and consider the partition

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

where Σ_{11} is $k \times k$ and Σ_{22} is $(p-k) \times (p-k)$. Prove the following statements.

   (a) $\text{null}(\Sigma_{22}) \subseteq \text{null}(\Sigma_{12})$, where $\text{null}(\cdot)$ denotes the null space of a matrix argument.

   (b) $\text{col}(\Sigma_{21}) \subseteq \text{col}(\Sigma_{22})$, where $\text{col}(\cdot)$ denotes the column space of a matrix argument.

   (c) If $X \sim \text{N}_p(\mu, \Sigma)$ then $X_1 - \Sigma_{12}\Sigma_{22}^g X_2$ is independent of X_2, where μ is a p-dimensional column vector, $X = (X_1', X_2')'$ is a partition of the appropriate dimensions, and Σ_{22}^g is a generalized inverse of Σ_{22}.

2. Let S be a nonempty subset of an inner product space V. The orthogonal complement to the set S is defined as

$$S^\perp := \{x \in V : \langle x, y \rangle = 0 \text{ for every } y \in S\}.$$

   (a) Show that $S^\perp$ is a subspace of V for any $S \subseteq V$.

   (b) Let $W \subseteq V$ be a finite dimensional subspace, and let $y \in V$. Show that there exist **unique** vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$.

   (c) Let $X \in \mathbb{R}^{n \times p}$. Verify that $\text{col}(X)$ and $\text{null}(X')$ are orthogonal complements.

3. Denote by W a matrix with $\text{col}(W) = \text{null}(P')$. Show that $\text{null}(W') = \text{col}(P)$.

4. Show that a $p \times p$ matrix A is symmetric and idempotent with rank s if and only if there exists a $p \times s$ matrix G with orthonormal columns such that $A = GG'$. Note that G is called a *semi-orthogonal* matrix.

5. For matrices $A \in \mathbb{R}^{p \times q}$, the *spectral* norm is defined as,

$$\|A\|_2 := \sqrt{\sup_{x \neq 0} \frac{x'A'Ax}{x'x}}.$$

Further, the eigenvalues of $A'A$ are the squares of the *singular values* of A, so sometimes the definition of the spectral norm is expressed as

$$\|A\|_2 := \sigma_{\max}(A),$$

where $\sigma_{\max}$ denotes the largest singular value of A.

(a) Verify that the spectral norm is a norm. Recall that a norm must satisfy the following axioms for any $A, B, C \in \mathbb{R}^{p \times q}$ and any $\alpha \in \mathbb{R}$.

i. $\|\alpha A\| = |\alpha| \|A\|$

ii. $\|A + B\| \leq \|A\| + \|B\|$

iii. $\|A\| \geq 0$ with equality if and only if $A = 0$.

(b) Show that the spectral norm is sub-multiplicative for square matrices. That is, for $A, B \in \mathbb{R}^{p \times p}$, $\|AB\|_2 \leq \|A\|_2 \|B\|_2$.

6. Suppose that the $m \times n$ matrix A has the form

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where A_1 is an $n \times n$ nonsingular matrix, and $m > n$. Define $A^+ := (A'A)^{-1}A'$, and prove that $\|A^+\|_2 \leq \|A_1^{-1}\|_2$.