ST 705 Linear models and variance components Homework problem set 4

January 28, 2025

- 1. Monahan problem 2.3.
- 2. Monahan problem 2.11.
- 3. Let S be a nonempty subset of an inner product space V. The orthogonal complement to the set S is defined as

$$S^{\perp} := \{ x \in V : \langle x, y \rangle = 0 \text{ for every } y \in S \}.$$

- (a) Show that S^{\perp} is a subspace of V for any $S \subseteq V$.
- (b) Let $W \subseteq V$ be a finite dimensional subspace, and let $y \in V$. Show that there exist **unique** vectors $u \in W$ and $z \in W^{\perp}$ such that y = u + z.
- (c) Let $X \in \mathbb{R}^{n \times p}$. Verify that col(X) and null(X') are orthogonal complements.
- 4. Prove that for any $A \in \mathbb{R}^{n \times p}$,

$$\{G : AGA = A\} = \{G + uv' : AGA = A, u \in \text{null}(A), \text{ and } v \in \text{null}(A')\}.$$

- 5. (a) Construct a counter example to show that A^g may not be symmetric, even if A is symmetric (e.g., even if A = X'X). That is, show that there exists A^g such that $[A^g]' \neq [A']^g$ for some symmetric matrix A.
 - (b) Prove that if A is symmetric, then $\frac{1}{2}(A^g + [A^g]')$ is a symmetric generalized inverse of A.
- 6. Let $A \in \mathbb{R}^{n \times p}$.
 - (a) Prove that if A^g is a generalized inverse of A (i.e., only satisfying $AA^gA = A$), then $(A^g)'$ is a generalized inverse of A'. Conclude from this fact that $P_X := X(X'X)^g X'$ is symmetric.
 - (b) Prove the existence **and** uniqueness of the Moore-Penrose generalized inverse, usually denoted A^+ , of A.

- (c) Show that if A has full column rank, then $A^+ = (A'A)^{-1}A'$.
- (d) Show that if A has full row rank, then $A^+ = A'(AA')^{-1}$.
- 7. Suppose you do not know that the rank of a matrix is equal to the number of nonzero singular values. Show that the rank of a projection matrix is equal to its trace. First think about how to show this in the symmetric case, and then consider the more general case of a non-symmetric idempotent matrix.