ST 705 Linear models and variance components
Homework problem set 9

March 26, 2024

1. Monahan exercise 3.26.

2. Consider the restricted linear model $Y = X\beta + U$ over the constrained parameter space $\{P'\beta = \delta\}$, for some full-column rank matrix P. Set up the Langrangian function and derive the *restricted normal equations* (RNE),

$$\begin{pmatrix} X'X & P \\ P' & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \theta \end{pmatrix} = \begin{pmatrix} X'y \\ \delta \end{pmatrix}.$$

3. Monahan exercise 4.2.

4. Suppose that $Y_i \sim \text{Binomial}(p, n_i)$ for $i \in \{1, \ldots, N\}$, and assume that $Y_1, \ldots, Y_N$ are independent.

   (a) Write this as a linear model.

   (b) Find the BLUE of p.

   (c) Find the MLE of p. How does the variance of the MLE compare to the variance of the BLUE?

5. The problem of least squares regression can be understood as a special case of the more general problem of ridge regression. For an n-dimensional column vector y and an $n \times p$ design matrix X, the problem of ridge regression is to solve for the parameter vector b that minimizes

$$a\|b\|_2^2 + \|y - Xb\|_2^2,$$

   where $a \geq 0$ is fixed.

   (a) Derive a closed-form expression of the ridge regression solution.

   (b) Assume that X has full column rank, and suppose that y is an observed instance of the random vector $Y = X\beta + U$, where $\beta \in \mathbb{R}^p$ is fixed and U satisfies the Gauss-Markov assumptions. Under what condition(s) is the ridge regression solution the BLUE for any β?

6. Suppose that $Y_1, \dots, Y_n \overset{\text{iid}}{\sim} \text{Uniform}(0, 2\theta)$, and define $U_i := Y_i - \theta$ for $i \in \{1, \dots, n\}$.

   (a) Find the mean and variance of $U := (U_1, \dots, U_n)'$.

   (b) Show that $Y := (Y_1, \dots, Y_n)'$ is generated according to a linear model that satisfies the Gauss-Markov assumptions.

   (c) Find the BLUE of θ, and denote the BLUE by $\hat{\theta}_{\text{OLS}}$.

   (d) Find c so that the estimator $\hat{\theta} = cY_{(n)}$ is unbiased for θ, where $Y_{(i)}$ denotes the ith order statistic, and compute the variance of $\hat{\theta}$.

   (e) Compare the variances of $\hat{\theta}_{\text{OLS}}$ and $\hat{\theta}$, and provide intuition for your finding.