ST 705 Linear models and variance components

Lab practice problem set 12

April 16, 2024

1. Let $X \sim \mathrm{N}_p(\mu, \Sigma)$. Show that for any partition of components, i.e.,

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}, \qquad \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{1m} \\ \vdots & \ddots & \vdots \\ \Sigma_{m1} & \cdots & \Sigma_{mm} \end{pmatrix},$$

   $X_1, \ldots, X_m$ are mutually independent if and only if $\Sigma_{ij} = 0$ for every $i \neq j$.

2. Suppose that (X, Y) has a bivariate distribution (**not necessarily Gaussian**) with mean $(\mu_X, \mu_Y)'$ and covariance matrix

$$\begin{pmatrix} \sigma_X^2 & \sigma_{X,Y} \\ \sigma_{Y,X} & \sigma_Y^2 \end{pmatrix}.$$

   (a) Show that if $E(Y \mid X) = \beta_0 + \beta_1 X$, then $\beta_1 = \sigma_{Y,X}/\sigma_X^2$ and $\beta_0 = \mu_Y - \beta_1 \mu_X$.

   (b) Show that if $E(Y \mid X) = \beta_0 + \beta_1 X$ and $\mathrm{Var}(Y \mid X) = \tau^2$, then $\tau^2 = \sigma_Y^2 - \sigma_{Y,X}^2/\sigma_X^2$.