

ST 705 MIDTERM

7 March 2024

NAME:

STUDENT ID:

- You have **75 minutes** to complete this exam.

- This is a **closed book, closed notes** exam.

- (3 points) Let $B \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^{p \times q}$. Show that if $\text{rank}(BC) = \text{rank}(B)$, then $\text{col}(BC) = \text{col}(B)$, where $\text{col}(\cdot)$ denotes the column space.
- (3 points) Show that the R^2 value for a simple linear regression can never achieve 1 if the observed data consists of repeated (different) observations of the response, y , at the same value of the predictor, x .
- Prove the following (related) statements.

(a) (3 points) For any $a \in \mathbb{R}^p$,

$$\|a\|_2 = \sup_{\|b\|_2=1} a'b.$$

(b) (3 points) For any $A \in \mathbb{R}^{n \times p}$,

$$\sigma_{\min}(A) = \inf_{\|x\|_2=1} \sup_{\|y\|_2=1} y'Ax,$$

where $\sigma_{\min}(A)$ denotes the smallest singular value of A .

- (3 points) Suppose that there exists a solution to the system of equations $Ax = c$. Then the general form of a solution is

$$x_z = Gc + (I - GA)z,$$

where z is an arbitrary vector of appropriate dimension and $G := (A'A)^g A'$ (do NOT need to show).

Find the z that minimizes the Euclidean norm of x_z .

- (3 points) Prove that for any $A \in \mathbb{R}^{n \times p}$,

$$\{G : AGA = A\} = \{G + uv' : AGA = A, u \in \text{null}(A), \text{ and } v \in \text{null}(A')\}.$$